The Sphere in Three Dimensions and Higher:
Generalizations and Special Cases

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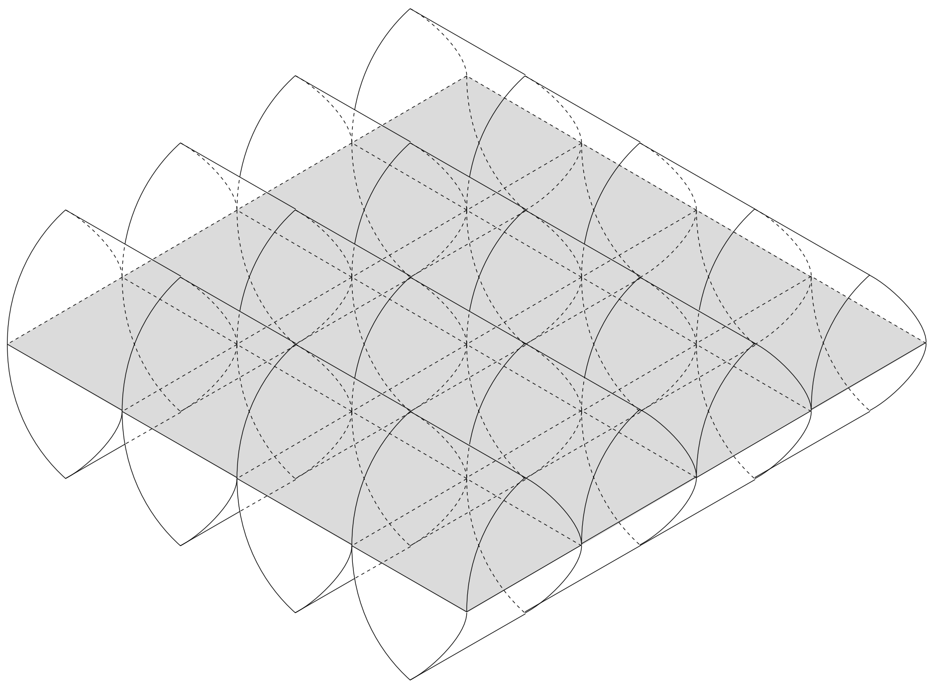
Abstract

I examine the general mathematical object that is the sphere. First, I examine several constructions general to all spheres. Then, I look at the Clifford Tori, the Hopf Fibration, group structure and surgery in $\mathbb{S}^3$, with respect to the unique structure of $\mathbb{S}^3$. Finally, I examine a unique dissection of $\mathbb{S}^3$ which takes advantage of a certain triangulation and the Clifford Tori. This dissection can be used to explain surgery on the unknot in an interesting way.



1 Overview of the Sphere

1.1 Definitions

1.1.1 Definition of the Sphere

The n-dimensional sphere is the set of all points in $n+1$ space at a given radius from a center point. The radius is denoted by r and the center by $\vec{\mathbf{c}}$.

$$\mathbb{S}^n = \left\{ \vec{\mathbf{x}} \in \mathbb{R}^{n+1} \mid \sqrt{\sum_{i=1}^{n+1} (x_i - c_i)^2} = r \right\}$$

Unless otherwise stated, the center of the sphere is assumed to be $\vec{\mathbf{0}}$ and the radius is assumed to be 1. Under this definition, $\mathbb{S}^0$ is the two points $\{1, -1\}$ in $\mathbb{R}^1$, $\mathbb{S}^1$ is the unit circle in $\mathbb{R}^2$, and $\mathbb{S}^2$ is the object which is commonly called the sphere or globe, in $\mathbb{S}^3$.

1.1.2 Definition of the N-Disk or N-Ball

An n-Disk is simply the set of all points inside $\mathbb{S}^{n-1}$, or all points less than or equal to r units away from the center $\vec{\mathbf{c}}$.

$$\mathrm{D}^n = \left\{ \vec{\mathbf{x}} \in \mathbb{R}^{n} \mid \sqrt{\sum_{i=1}^{n} (x_i - c_i)^2} \leq r \right\}$$

Under this definition, the 1-disk is the line segment between r and $-r$ on $\mathbb{R}^1$, the 2-disk is the interior of the circle of radius r in $\mathbb{R}^2$, and the 3-disk or 3-ball is the interior of the sphere of radius r in $\mathbb{R}^3$. As above, the center is assumed to be $\vec{\mathbf{0}}$ and the radius to be 1 unless otherwise specified.

1.1.3 Definition of a Manifold

Throughout this paper I will use the term manifold in its standard topological sense. A n-dimensional manifold is usually defined as a set with a collection of patches, or 1-1 functions from $D \to M$, where D is an open subset of $\mathbb{R}^n$, such that the following two conditions are satisfied:

(i.) The images of the patches cover M

(ii.) For two patches τ and σ, the composition $\tau^{-1}\sigma$ and $\sigma^{-1}\tau$ are differentiable and defined on open sets in $\mathbb{R}^n$ (O'Neill).

1.1.4 Definition of $\times$, or Cartesian product

The $\times$ operator is the Cartesian product. It takes two manifolds of dimension n and m, and duplicates the one at every point along the other, creating a manifold of dimension $m+n$.

1.1.5 Definition of Generator

A generator is a manifold M contained in another manifold P such that M crossed with some other manifold N makes P. In other words, if $M \times N = P$, then M and N are generators of P. For example, since $\mathbf{T}^2 = \mathbf{T}^1 \times \mathbf{T}^1$, the generators of $\mathbf{T}^2$ are the two $\mathbf{T}^1$.

1.1.6 Definition of Equator

An equator inside $\mathbb{S}^n$ is a $\mathbb{S}^{n-1}$. This can be seen by setting one of the coordinates, say x_{n+1}, to zero, and taking all remaining points in $\mathbb{S}^n$.

$$\left\{\vec{\mathbf{x}} \in \mathbb{R}^{n+1} \mid \sqrt{\left(\sum_{i=1}^{n} x_i{}^2\right) + 0^2} = r\right\} = \left\{\vec{\mathbf{x}} \in \mathbb{R}^n \middle| \sqrt{\sum_{i=1}^{n} x_i{}^2} = r\right\} = \mathbb{S}^{n-1}$$

1.2 Construction of $\mathbb{S}^n$ by two N-disks

It's fairly trivial to see that two line segments, each end of one segment joined to an end of the other segment, make a circle (or something that can be deformed into a circle). However, the process generalizes to all dimensions, and provides us with a technique of visualizing $\mathbb{S}^n$.

Two 2-disks, or filled circles, can be joined along their circular boundaries, to make $\mathbb{S}^2$. Each disk makes a hemisphere, and the boundary between the two becomes the equator.

In general, an equator as defined above is $\mathbb{S}^{n-1}$ in $\mathbb{S}^n$. Now, all the other points in $\mathbb{S}^n$ are the points where $x_n \neq 0$. So the other points in $\mathbb{S}^n$ are split into two groups: $x_n > 0$ and $x_n < 0$. We can see that these two sets of points are N-disks because we can let x_n range from 1 to 0, and get a corresponding $\mathbb{S}^{n-1}$ for each value of x_n.

2 Visualizing the N-Sphere

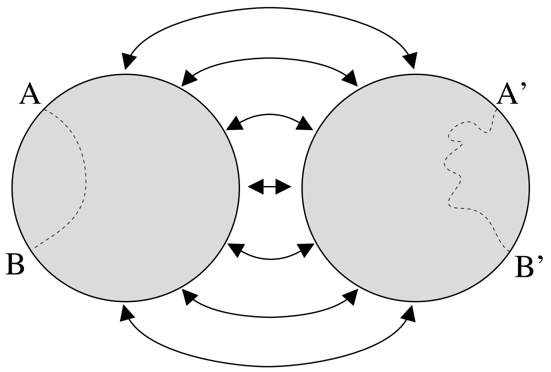
2.1 The two balls model

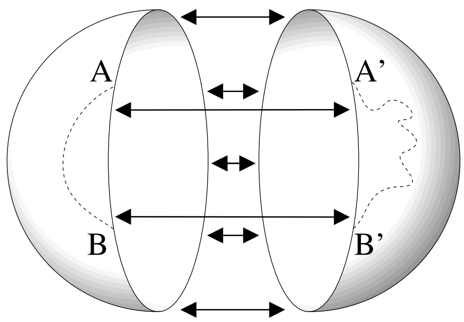
This observation about equatorial surfaces can be used as an aid to visualization as well.

2.1.1 The two balls model of $\mathbb{S}^2$

First, imagine that we were beings who inhabited a two-dimensional universe. Visualizing three-dimensional space would (probably) not come naturally to us, and thus neither would visualizing $\mathbb{S}^2$, which needs to be embedded in at least three dimensions.

However, we could visualize $\mathbb{S}^2$ by thinking of it as two disks, plus a special rule. The special rule goes like this: whenever you cross the boundary of one disk, you reappear at the corresponding point on the boundary of the other disk. Each disk is a hemisphere, and the boundary is the equator. Thought of this way, the rule becomes obvious: whenever you leave one hemisphere, and cross the equator, you reappear at the corresponding point in the other hemisphere.

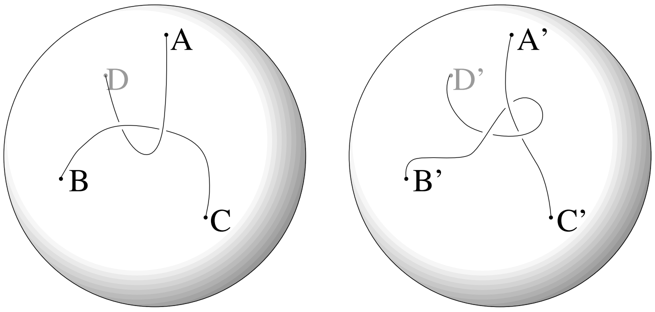




Note that when our two dimensional being lays the two disks flat out in $\mathbb{R}^2$, there are two ways that they can be laid down. First, both disks can simply be flattened onto $\mathbb{R}^2$. If this is done, then a vector pointing out of the sphere on the lower hemisphere will be pointing down from $\mathbb{R}^2$, and a vector pointing out of the sphere on the upper hemisphere will be pointing up on $\mathbb{R}^2$. Alternately, the bottom hemisphere can be reflected or flipped, so that a vector pointing out on the sphere will be pointing up from $\mathbb{R}^2$, on both hemispheres. Note that this act of flipping means that we will have to be extra careful about how we identify the edges of the hemisphere; specifically, points will be mapped to the corresponding point on the hemisphere *after the reflection.* For practical purposes, we will consider the first model where the lower hemisphere is *not* flipped.

2.1.2 The two balls model of $\mathbb{S}^3$

Now, imagine that we are beings who inhabit a three dimensional universe (Hopefully this is not too hard). Visualizing four-dimensional space would not come naturally to us, and neither would visualizing $\mathbb{S}^3$, which needs to be embedded in at least four dimensions. But we can visualize the hypersphere by thinking of it as two filled balls, or filled $\mathbb{S}^2$s, plus a special rule, just like in the two dimensional case.



Finally, similarly to the two disks model of $\mathbb{S}^2$, we can flatten the hemispheres onto $\mathbb{S}^3$ without flipping one, or with flipping one. Just as before, we will use the model without the flip.

2.1.3 Using a two balls model to visualize higher dimensional spheres

s If we wanted to visualize $\mathbb{S}^4$, we could first visualize $\mathbb{S}^3$ in this way, and then make two copies of $\mathbb{S}^3$, and consider each to be a $\mathbf{D}^4$ or filled $\mathbb{S}^3$. We could then glue each point in the $\mathbb{S}^3$ to a corresponding point in the other $\mathbb{S}^3$, and we would have $\mathbb{S}^4$. Unfortunately, this becomes a bit cumbersome and complicated for dimensions higher than three.

2.2 Stereographic Projection

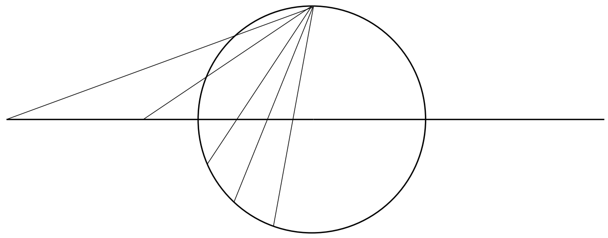
Stereographic projection is a way of projecting $\mathbb{S}^n$ onto $\mathbb{R}^n$. Intuitively, it can be thought of as the process of puncturing the sphere and stretching it out flat, onto Euclidean space. It too can be used as an aid to visualization.

2.2.1 Stereographic Projection of $\mathbb{S}^1$

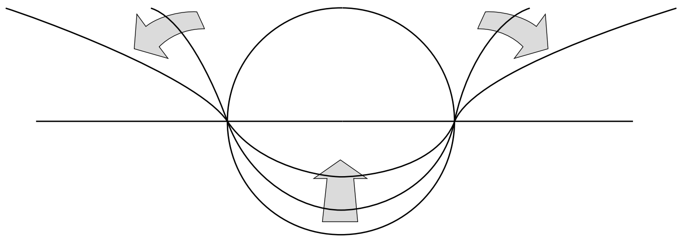
To stereographically project $\mathbb{S}^1$ onto $\mathbb{R}^1$, first choose an equator, and a "north pole." In the case of $\mathbb{S}^1$, the equator simply is two antipodal points on the circle, and the north pole is the point on the circle equally between them. For simplicity, let's assume the circle is the unit circle, the north pole is the point (0,1), and the equator the set of points $\{(1,0),(-1,0)\}$. We are now going to map, or stereographically project, the circle onto the x axis.

The map is as follows: Draw a line from the north pole, through a point p on the circle, and extend it until it hits the x-axis at the point p'. Stereographic projection is simply the map which takes each point p on the circle and maps it to the point p' in $\mathbb{R}^1$. In the case with $\mathbb{S}^1$, the map sends $p = (x, y) \in \mathbb{S}^2$ to $p' = \frac{x}{1-z} \in \mathbb{R}^1$.

Notice that every point on $\mathbb{S}^1$ has an image on $\mathbb{R}^1$, except for the north pole itself. The north pole is considered to be mapped to a point at infinity. So we can say that stereographic projection from $\mathbb{S}^1 \to \mathbb{R}^1 \cup \{\infty\}$ is a bijection.



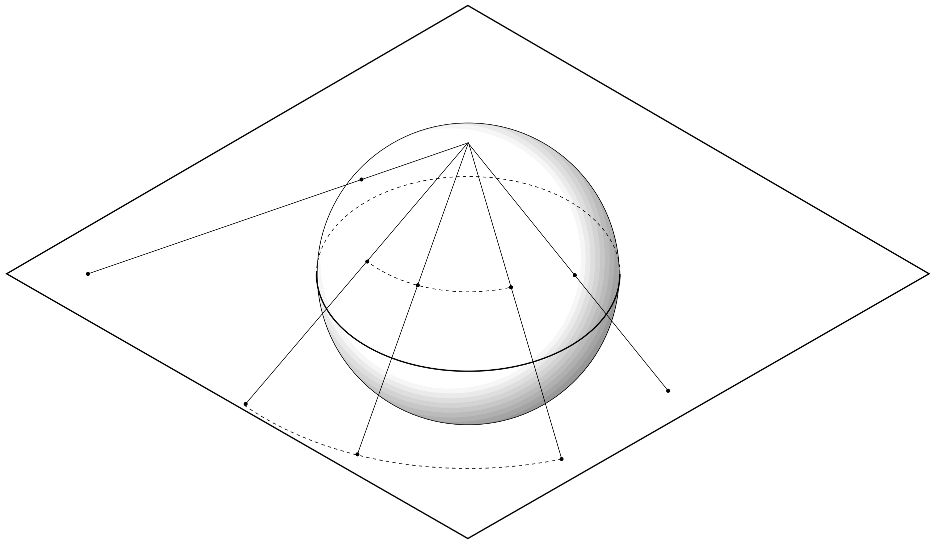
As mentioned above, stereographic projection can be thought of as puncturing the sphere, and stretching it out flat onto the correct Euclidean space. Points in the "southern hemisphere" are mapped to points between 1 and -1, and points in the "northern hemisphere" are mapped to points outside.



2.2.2 Stereographic Projection of $\mathbb{S}^2$

To stereographically project $\mathbb{S}^2$,imagine $\mathbb{S}^2$, centered at the origin. To project the sphere onto the x-y plane, we choose the "north pole" to be the point $(0,0,1)$. Then, to project a point $\vec{\mathbf{p}} = (x,y,z) \in \mathbb{S}^2$ onto the x-y plane, we draw a line from the north pole, through $\vec{\mathbf{p}}$ to it's image point $\vec{\mathbf{p}}'$ on the x-y plane.

$\vec{\mathbf{p}} = (x,y,z) \in \mathbb{S}^2 \mapsto \left(\frac{x}{z-1}, \frac{y}{z-1}\right) \in \mathbb{R}^2$



Again, the north pole is mapped to the point at infinity. Now the hypothetical two-dimensional beings have another way to visualize $\mathbb{S}^2$: they can think of it as regular $\mathbb{R}^2$, plus a point at infinity. If they were moving, say, a little square around the sphere, and watching how it's image changed in the plane as it moved, they would see it get larger as it got further from the south pole, which is mapped to the the origin on the plane. If they moved the square from the south pole, through the north pole, and back to the south pole, the two dimensional beings would see the square get larger and larger as it approached the north pole. When the north pole was inside the square, they would see the square all around them. As the square moved back to the south pole, they would see the square approaching from the *other* direction, getting smaller as it approached.

2.2.3 Stereographic Projection in $\mathbb{S}^n$

Stereographic projection is possible in general, for $\mathbb{S}^n$. The map is:

$$\vec{\mathbf{p}} = (x_1, x_2, ..., x_{n+1}) \in \mathbb{S}^n \mapsto \left(\frac{x_1}{x_{n+1}-1}, \frac{x_2}{x_{n+1}-1}, ..., \frac{x_n}{x_{n+1}-1}\right) \in \mathbb{R}^n$$

Now we have another way to visualize the three sphere: we can think of it as $\mathbb{R}^3$, plus a point at infinity. If we were moving a cube around the three sphere, and looking at it's image in Euclidean three-space under stereographic projection, it would get bigger as it got further from the south pole, which is the origin in $\mathbb{R}^3$. Like the image of $\mathbb{S}^2$ in $\mathbb{R}^2$, if we were moving a cube along a great circle from the south pole, through the north pole, back to the south pole, we would see it start out small at the origin, get larger as it went further from the origin. When the north pole crossed inside the little cube, we would see the cube all around us. Then, as the square passed back towards the south pole, we would see it shrink again as it approached.

Stereographic projection offers another technique of visualizing or thinking about higher dimensional spheres, and another point of view on $\mathbb{S}^3$.

3 Inversion

3.1 Inverting $\mathbb{S}^{n-1}$ inside $\mathbb{S}^n$

From our two balls model, it is easy to see that we can shrink the equatorial $\mathbb{S}^{n-1}$ inside $\mathbb{S}^n$, until it is a small sphere surrounding the north pole or the south pole.

Let $\vec{\mathbf{x}} \in \mathbb{S}^{n-1}$, and let $P(\vec{\mathbf{x}}) : \mathbb{S}^{n+1} \to \mathbb{R}^{n-1}$ such that $\vec{\mathbf{x}} \mapsto (\vec{\mathbf{x}} \sin\theta, \cos\theta)$. Then varying θ in the range $-1 \leq \theta \leq 1$ gives a continuous family of $\mathbb{S}^{n-1}$ inside $\mathbb{S}^n$. When $\theta = 1$ or -1, the sphere is shrunk to a single point. When θ is close to 1 or -1, it's a little sphere around either the north or south pole. When θ is near zero, it is nearly equatorial.

If we now consider the two regions into which the $\mathbb{S}^n$ is divided by this moving $\mathbb{S}^{n-1}$, this might seem very similar to some process of turning the $\mathbb{S}^{n-1}$ "inside out." However, this is not quite correct. We might look at the $\mathbb{S}^{n-1}$ inside $\mathbb{S}^n$ at some point and label the two regions as "inside" and "outside" based on which region was smaller. We do this because we are used to Euclidean space, where the "inside," of a closed region usually appears smaller, and the outside usually includes the point at infinity.

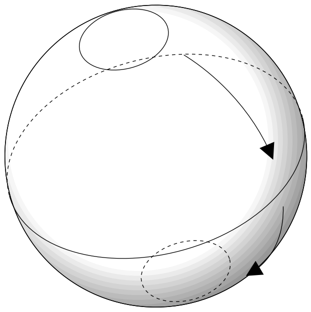
However, the region inside the sphere is already finite, and so calling this process "inverting," or swapping the "inside" and "outside" of $\mathbb{S}^{n-1}$ inside $\mathbb{S}^n$ really doesn't make a whole lot of sense. All we can say is that we can distort $\mathbb{S}^{n-1}$ inside $\mathbb{S}^n$ in such a way as to shrink one region down to a point while expanding another region from a point to the whole sphere.

Following from this, we see that any closed surface inside $\mathbb{S}^n$ can be "inverted," by simply gluing $\mathbb{S}^{n-1}$ to a small region of the manifold, and then "inverting" this sphere.

From now on I'll use the word "inversion" in the sphere to mean a process like this which apparently reverses the two regions in $\mathbb{S}^n$ created by a dividing surface in $\mathbb{S}^n$.

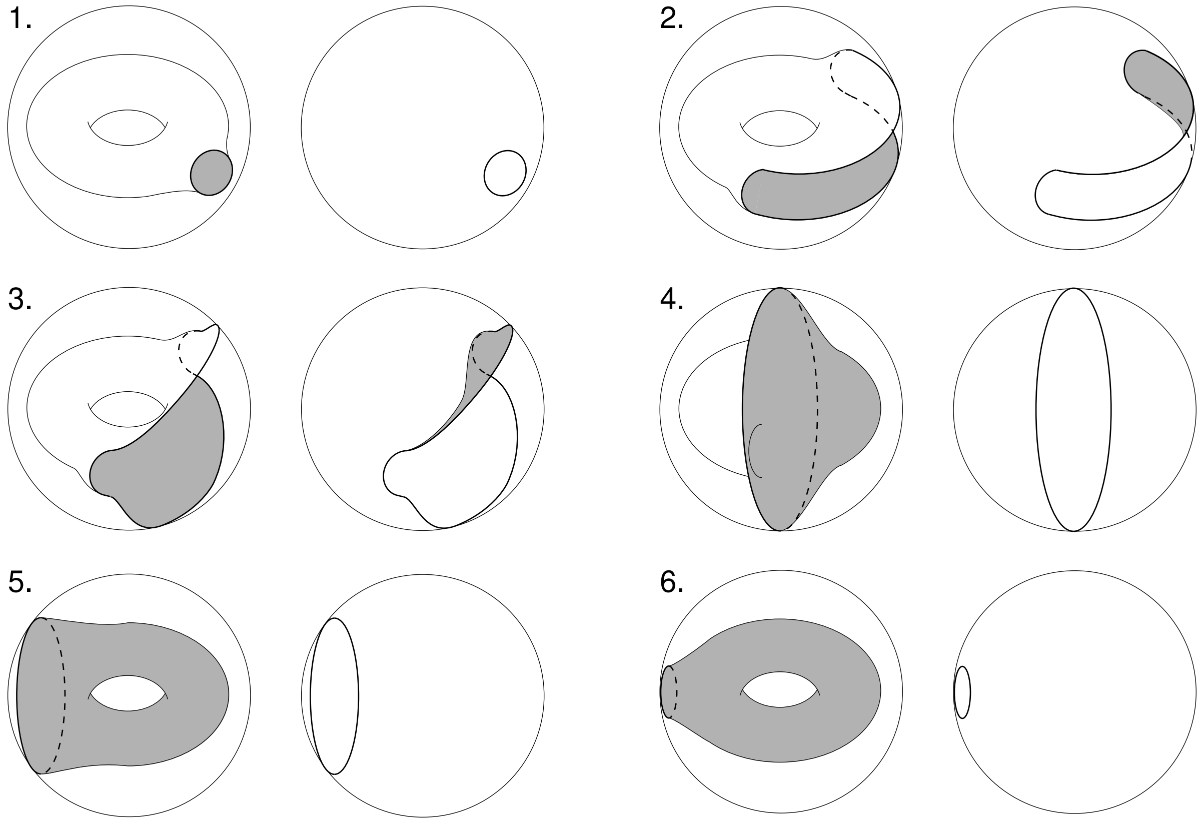
3.2 Inverting $\mathbf{T}^{n-1}$ inside $\mathbb{S}^n$

Notice that $\mathbf{T}^1$ and $\mathbb{S}^1$ are identical manifolds. So we know, trivially, that we can invert $\mathbf{T}^1$ inside $\mathbb{S}^2$.



Notice also that we can embed the flat torus, $\mathbf{T}^2$, inside $\mathbb{S}^3$. To see this best, consider our space to be $\mathbb{C}^2 = \mathbb{R}^4$. Then the sphere consists of all points (z_1, z_2) such that $\sqrt{|z_1| + |z_2|} = 1$, and the torus consists of all points (z_1, z_2) such that $|z_1| = |z_2| = \frac{1}{\sqrt{2}}$.

Inside $\mathbb{S}^3$, this torus can be "inverted," that is, it can be deformed in such a way so that (1) the generators of the torus are swapped, and (2) the apparent "outside" is swapped with the apparent "inside." Here are a series of pictures to illustrate the second point; pictures showing the generators being swapped are in section 4.2.



The important point to notice from this is that $\mathbf{T}^2$ divides $\mathbb{S}^3$ into two regions, both of which are topologically equivalent to a filled $\mathbf{T}^2$. It is by no means obvious that this is a general fact about $\mathbf{T}^{n-1}$ inside $\mathbb{S}^n$. Furthermore, we can ask whether there is any deeper structure behind this inversion, or whether it is merely a manifestation of the fact, noted above, that any manifold can be inverted inside of the sphere.

3.2.1 Inverting $\mathbf{T}^{n-1}$ inside $\mathbb{S}^n$

We know that any manifold can be inverted inside $\mathbb{S}^n$ by gluing it to a sphere. If we know that $\mathbf{T}^{n-1}$ in general can be embedded inside $\mathbb{S}^n$, then we will know that $\mathbf{T}^{n-1}$ can be inverted inside $\mathbb{S}^n$, by gluing the torus to a $\mathbb{S}^{n-1}$ and inverting that $\mathbb{S}^{n-1}$.

The proof is inductive. To begin with, we know that $\mathbf{T}^2$ can be embedded in $\mathbb{R}^3$, which can be stereographically projected to $\mathbb{S}^3$.

If, in general, $\mathbf{T}^{n-1}$ can be embedded in $\mathbb{R}^n$, then we can embed $\mathbf{T}^{n-1} \times I$ in $\mathbb{R}^n$. Think of this simply as as a "thickened" or "fattened" torus. Next, we cross both with I, to get $\mathbf{T}^{n-1} \times I \times I \subset \mathbb{R}^n \times I$. Now, we can draw a little circle in $I \times I$ to see that $\mathbf{T}^n \subset \mathbf{T}^{n-1} \times I \times I$. And, of course, $\mathbb{R}^n \times I \subset \mathbb{R}^{n+1}$. So

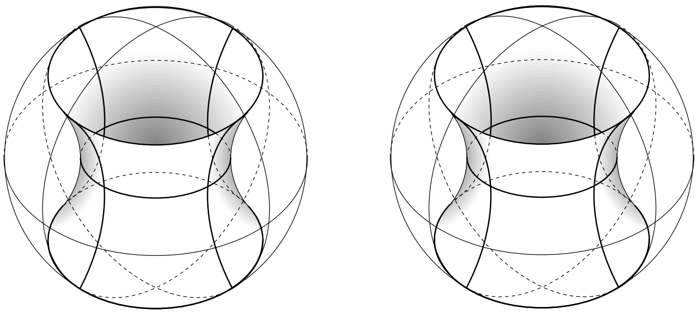
$\mathbf{T}^{n-1} \subset \mathbb{R}^n \implies \mathbf{T}^n \subset \mathbb{R}^{n+1}$. Since $\mathbf{T}^n$ can be embedded in $\mathbb{R}^{n+1}$, $\mathbf{T}^n$ can be embedded in $\mathbb{S}^{n+1}$ as we can just project $\mathbb{R}^{n+1}$ to $\mathbb{S}^{n+1}$ via stereographic projection.

So, in general, $\mathbf{T}^n$ can be embedded and "inverted" inside $\mathbb{S}^n$. The next question is whether the neat symmetry of $\mathbf{T}^2$ in $\mathbb{S}^3$ generalizes as well. To do this we will need to look closer at the structure of $\mathbb{S}^3$.

4 Clifford Tori and Hopf Fibration

4.1 The Clifford Tori

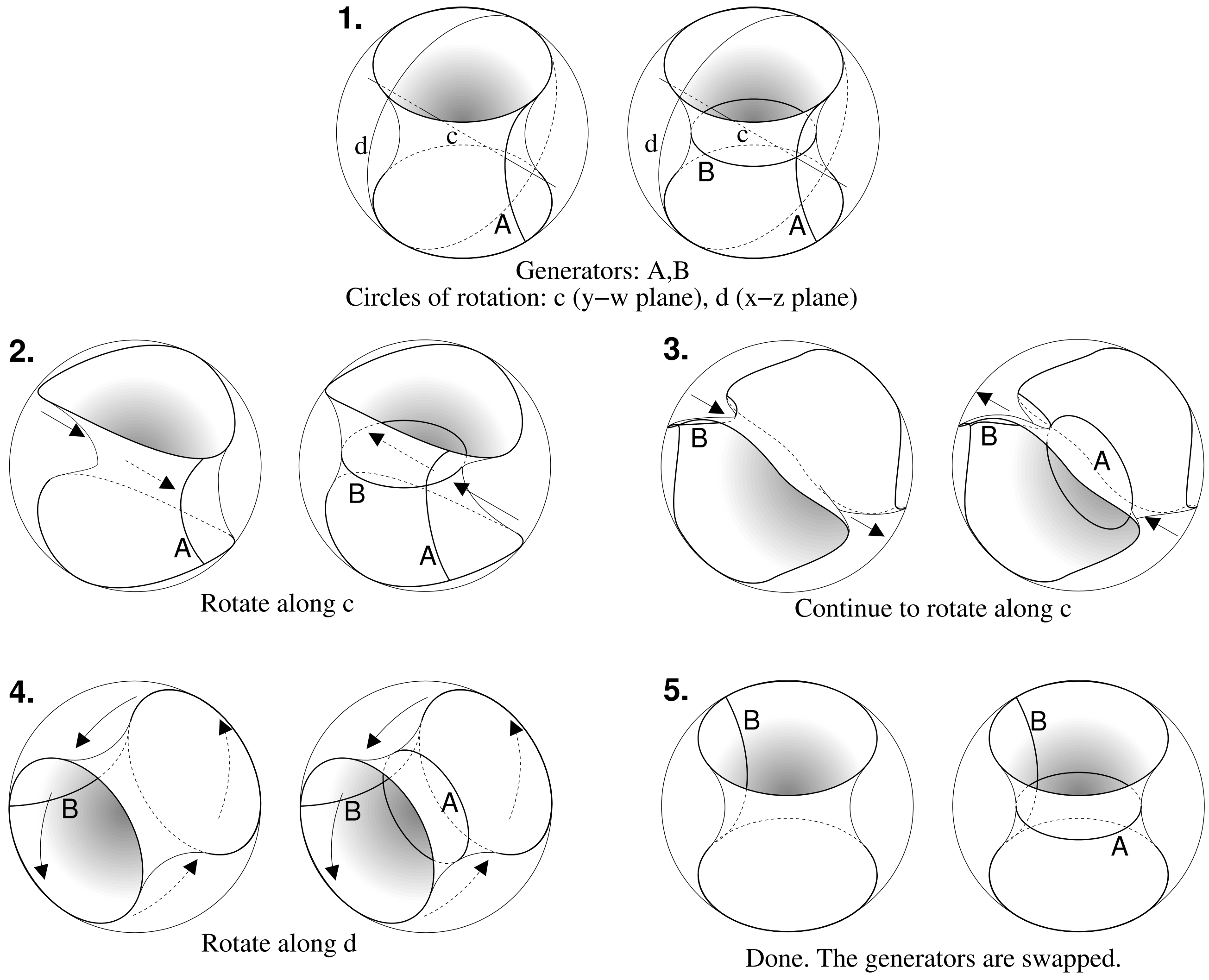
Two unit circles in $\mathbb{S}^3$, one in the x-y plane and the other in the z-w plane, form the basis of a curious structure on $\mathbb{S}^3$. The surface of a tubular neighborhood a fixed distance from either of these circles forms a torus in $\mathbb{S}^3$. The Clifford Tori are the family of tori generated by tubular neighborhoods around either of these centerlines. A tubular neighborhood at a distance of n from one centerline is the tubular neighborhood at a distance of $1-n$ from the other, where $0 \leq n \leq 1$. The family of tori generated by varying n are called the Clifford Tori. The tubular neighborhood at a distance of $\frac{1}{2}$ forms the flat torus in $\mathbb{S}^3$ mentioned above. I'll call this torus the equatorial torus, because, like an equator, we shall see that it divides the sphere neatly into two identical pieces.



It is important to note that $\mathbf{T}^2$ can be embedded in $\mathbb{R}^4$ so that it is "flat," that is, so that it's Gaussian curvature is 0 at every point. This is not possible in $\mathbb{R}^3$. Additionally, the Clifford Tori are embedded "flatly" in $\mathbb{S}^3 \subset \mathbb{R}^4$.

4.2 Inversion of $\mathbf{T}^2$ in $\mathbb{S}^3$ by isometries

Not only does $\mathbf{T}^2$ divide $\mathbb{S}^3$ into two filled tori, but this division can be done in a geometrical way. In other words, if we cut the sphere into two pieces along this special torus, we could swap the two pieces (in $\mathbb{R}^4$) of the $\mathbb{S}^3$ and rearrange them back into a sphere. This means we might expect it to be possible to "invert" or swap these regions in $\mathbb{S}^3$ via isometries. In fact, this is possible. We can swap the two centerlines, (and their tubular neighborhoods along with them) by rotating first by $\frac{\pi}{2}$ in the x-z plane, and then by $\frac{\pi}{2}$ in the y-w plane.



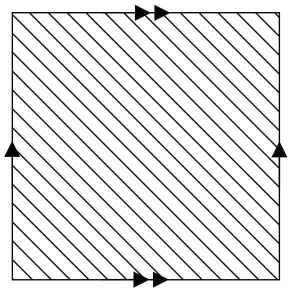
This operation also makes it possible to see another interesting fact about inverting $\mathbf{T}^2$ in $\mathbb{S}^3$. Since $\mathbf{T}^2 = \mathbf{T}^1 \times \mathbf{T}^1$, we can distinguish two unique, perpendicular lines on the surface of $\mathbf{T}^2$ (A and B) and call them the generators. If we choose the generators so that one is in the direction of one of the centerlines, and the other in the direction of the other centerline, and then watch these generators as we rotate the sphere, we can see that they are swapped as well.

So we see that this rotation of the sphere moves a point not on this equatorial torus from one filled torus to the other. It also maps the equatorial torus to itself. This can all be done in an explicit algebraic manner using the fact that the group $SU(2)$ acts on $\mathbb{S}^(3)$ isometrically (see Vajk for details).

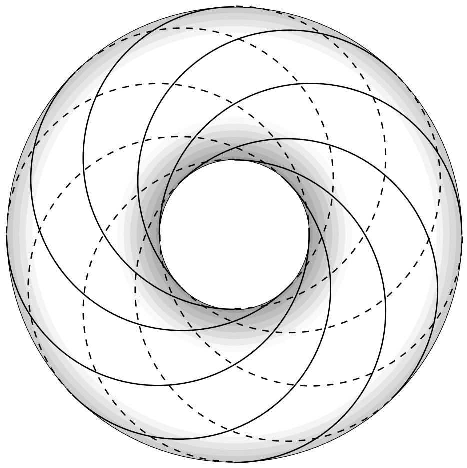
4.3 The Hopf Fibration

Now that we have all of this machinery, we can consider another curious property of $\mathbf{T}^2$, the Hopf fibration. First we will consider a fibration on $\mathbf{T}^2$, and then examine what it looks like when this fibration is extended to the Clifford Tori.

We can think of the $\mathbf{T}^2$ as a square, or $I \times I$, with edges identified. Each pair of parallel and identified edges becomes one of the generators. First, draw a diagonal from one corner to it's opposite. Now, draw a series of lines parallel to that diagonal.



Now consider what these lines become when the square is wrapped up into a torus. They become a series of circles, each one circling the torus exactly once in the direction of one generator, and once in the direction of the other generator. Notice that the circles are linked, each one passing through each other one.



Now, consider if each of the Clifford Tori had this fibration on them. This is the Hopf Fibration. As the tubular neighborhood of these centerlines is shrunk, the tori get smaller, and the linked circles more closely approximate one of the centerlines.

4.4 Generalizing the Clifford Tori and The Hopf Fibration

In light of our knowledge of other general processes in $\mathbb{S}^n$, it now makes sense to ask whether the Clifford Tori and the Hopf Fibration can be generalized into arbitrary dimension. To answer this we need to ask what is special about the Clifford Tori that creates this symmetric division. The special characteristic seems to be the two centerlines, the circles in the x-y and z-w planes.

The generalization of the Clifford Tori to arbitrary dimension looks like this. Inside $\mathbb{S}^n$, draw a unit circle, say in the x-y plane. Then draw a $\mathbb{S}^{n-2}$ in the remaining subspace of $\mathbb{S}^n$ where $x = y = 0$. For $\mathbb{S}^2$, this turns out to be $\mathbb{S}^0$, consisting of just two points, the north pole and the south pole. In $\mathbb{S}^3$, it is another unit circle in the z-w plane. And in $\mathbb{S}^4$ it is a sphere.

Now we can draw a surface inside the $\mathbb{S}^n$, by making a tubular neighborhood at a distance of $\frac{1}{2}$ around either our unit circle, or the $\mathbb{S}^{n-2}$. What do these regions look like when filled? Well, the region around the unit circle is $\mathbb{S}^1 \times D^{n-2}$. And the tubular neighborhood around $\mathbb{S}^{n-2}$ is $\mathbb{S}^{n-2} \times D^1$. Only in $\mathbb{S}^3$, when $n - 2 = 1$, are the two regions the same.

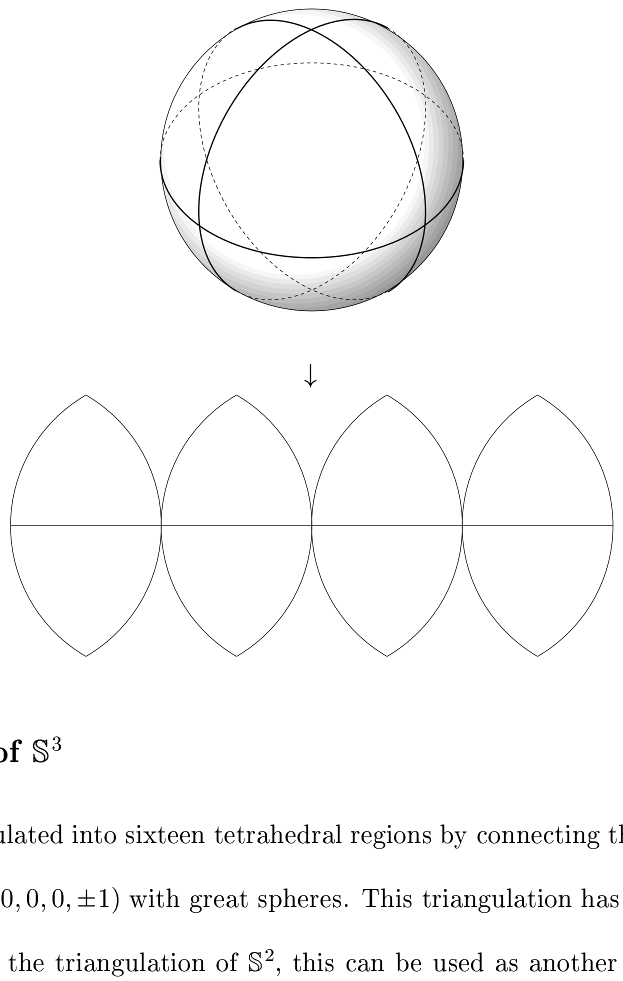
So the Clifford Tori can be generalized to all spheres, but only in $\mathbb{S}^3$ are they symmetric. It is not clear how the Hopf Fibration could be extended to the arbitrary dimension, as it relies on the two centerlines being the same manifold, which only happens in $\mathbb{S}^3$. In addition, we have found another surface which divides

$\mathbb{S}^n$, that is, $\mathbb{S}^1 \times \mathbb{S}^{n-2}$. But only in $\mathbb{S}^3$ does this manifold divide the sphere into two identical pieces, so the Clifford Tori and the Hopf Fibration are special properties of $\mathbb{S}^3$.

5 Triangulation of $\mathbb{S}^n$

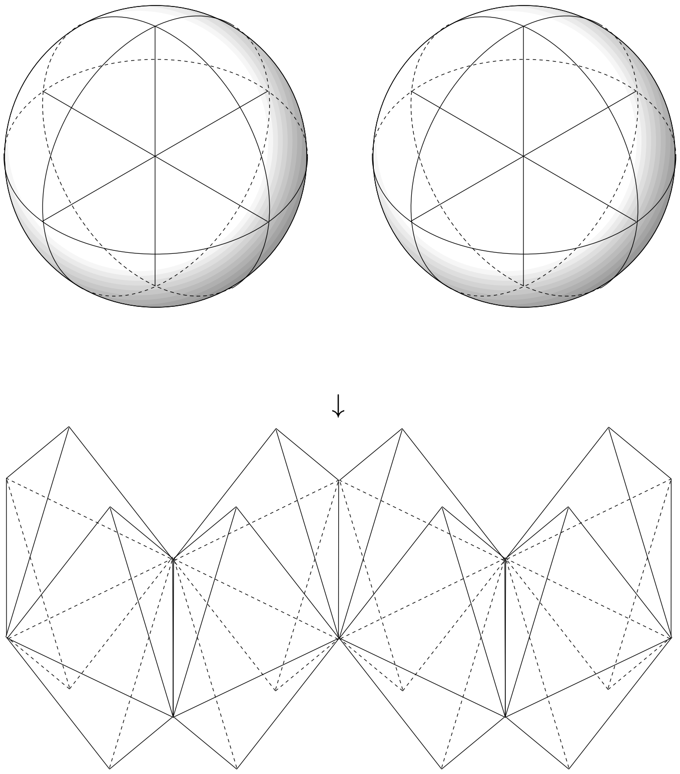
5.1 Triangulation of $\mathbb{S}^2$

$\mathbb{S}^2$ can be triangulated into eight triangular regions. This triangulation has the same symmetry group as the octahedron. The triangulation basically consists of connecting the six points $(\pm1, 0, 0)$, $(0, \pm1, 0)$, $(0, 0, \pm1)$ with great circles in the x-y, y-z and z-x planes. Cutting the sphere along these lines, and then laying out the eight resulting triangles in a plane can be thought of as another visualization technique for two-dimensional beings.



5.2 Triangulation of $\mathbb{S}^3$

Similarly, $\mathbb{S}^3$ can be triangulated into sixteen tetrahedral regions by connecting the eight points $(\pm1, 0, 0, 0)$, $(0, \pm1, 0, 0)$, $(0, 0, \pm1, 0,)$, $(0, 0, 0, \pm1)$ with great spheres. This triangulation has the same symmetry as the sixteen-cell polytope. Like the triangulation of $\mathbb{S}^2$, this can be used as another method of visualization of $\mathbb{S}^3$, and it will be, later.



Notice that there are a great many more possibilities for how these sixteen tetrahedra can be arranged than there are for the triangulation of $\mathbb{S}^2$. Might there be a particularly orderly arrangement for them, that had no analogue in $\mathbb{S}^2$?

5.3 Triangulation of $\mathbb{S}^n$

Since an analogue of the symmetry group of the octahedron and the sixteen-cell exists in every dimension, we can generalize this process of triangulation to $\mathbb{S}^n$, decomposing it into 2^{n+1} n-cells.

6 Surgery on Knots

Surgery on knots is the process of embedding a knot inside a manifold, removing a tubular neighborhood of the knot from the manifold, and then re-gluing it to the manifold in a different fashion. Although it might not seem possible, different manifolds can be created in the process. For the purposes of this paper, I will look at one of the simplest of these processes; surgery on the unknot, or circle, in $\mathbb{S}^3$.

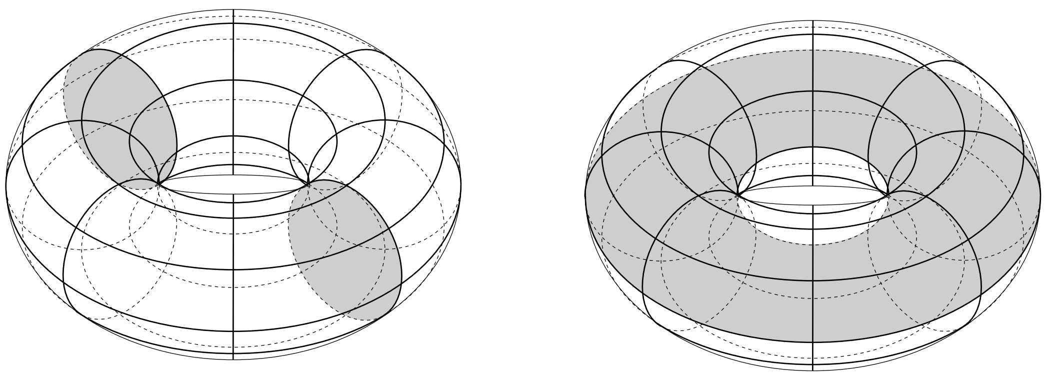
6.1 Turning $\mathbb{S}^3$ into $\mathbb{S}^2 \times \mathbb{S}^1$ by removing and replacing $\mathbf{T}^2$

When a tubular neighborhood of the unknot is removed from the sphere, and glued back to the sphere in a different fashion, the resulting manifold is $\mathbb{S}^2 \times \mathbb{S}^1$. This is not an easy fact to see.

6.1.1 Separate $\mathbb{S}^3$ into two filled $\mathbf{T}^2$

First, we must notice that removing a circular neighborhood of the unknot is the same as removing a filled torus. Since we already know that cutting $\mathbb{S}^3$ along the surface of $\mathbf{T}^2$ separates it into two filled tori, we know that removing one filled torus leaves another filled torus. So imagine our sphere is now separated into two identical filled tori.

Now, let's look at the surface of a great sphere, and how it is divided by the two tori. Look again at the picture of the equatorial torus in section 4.1. Consider the great sphere which is flattened into the x-y plane in these images. It is split into three pieces. It passes through one filled torus twice, in two disk-like regions around opposite poles of the great sphere. It passes through the other filled torus in an annular region.



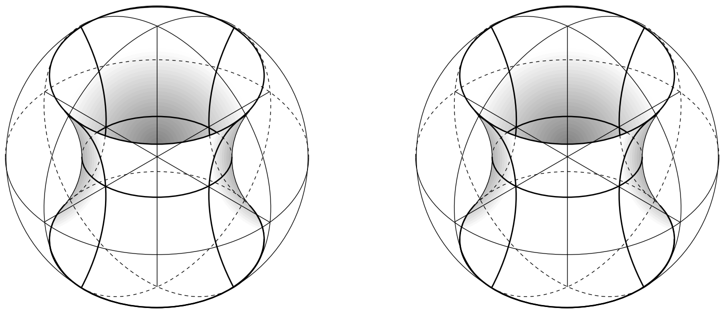
Now if we look at one of our two filled tori, say, the one on the right above, and draw in this annular surface formed by intersection with the great sphere, we can see how it must be glued to the other filled torus. In particular, this annular region must be glued to two disks in the other torus to make a $\mathbb{S}^2$. So the two tori are glued with generators swapped; the generator that "goes around the hole" in one torus is glued to the generator that "goes through the hole" in the other, and vice versa. In other words, the "tube" of one torus goes through the "hole" of the other.

6.1.2 Re-glue the tori

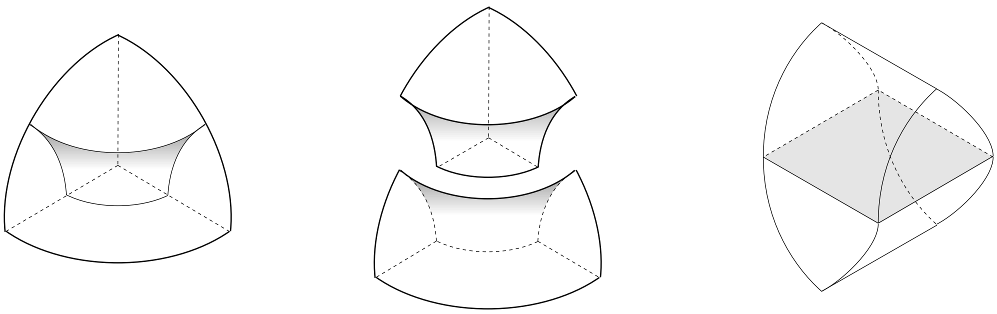
Now, if we re-glue the tori so that the generators are not swapped, and corresponding points on the tori are glued to each other. then we will get a different manifold. Each boundary of each annular region is glued to the corresponding boundary of the corresponding annular region in the other torus. And each circular region is glued to another circular region, to make a sphere. In other words, at each point on a circle going "around the hole" of the torus, there is a sphere formed by gluing two disks that intersects the circle at that point. So, after the surgery, the manifold is no longer $\mathbb{S}^3$, but instead $\mathbb{S}^1 \times \mathbb{S}^2$, another three dimensional manifold that can not be topologically distorted to the sphere.

7 A visualization of The Clifford Tori, Triangulation, and Surgery

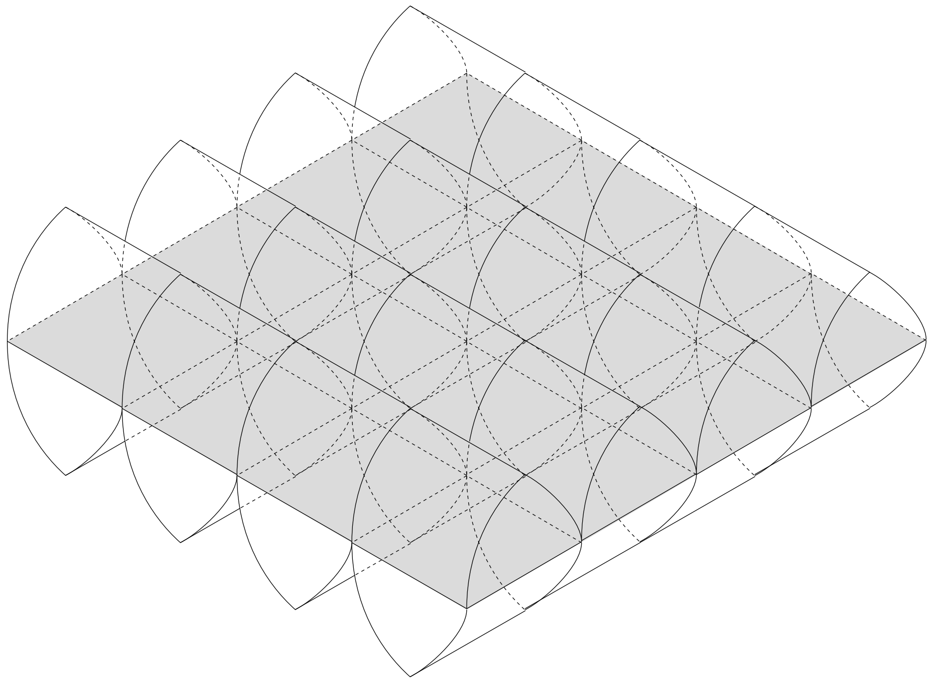
Consider this picture of $\mathbb{S}^3$ where both the triangulation into sixteen tetrahedral regions (section 5.2) and the equatorial torus (section 4.1) are visible.



Let us consider now just a single tetrahedral cell. By symmetry, we know that each tetrahedral cell is identical. $\frac{1}{16}$th of the equatorial torus is visible inside this tetrahedron. With a little examination, we can see that this section of the equatorial torus is a square, and we can also tell that it divides the tetrahedron into two similarly shaped triangular prisms. Now we can distort out tetrahedral cell until the section of the equatorial torus is a flat square and the two triangular prisms are the same shape.

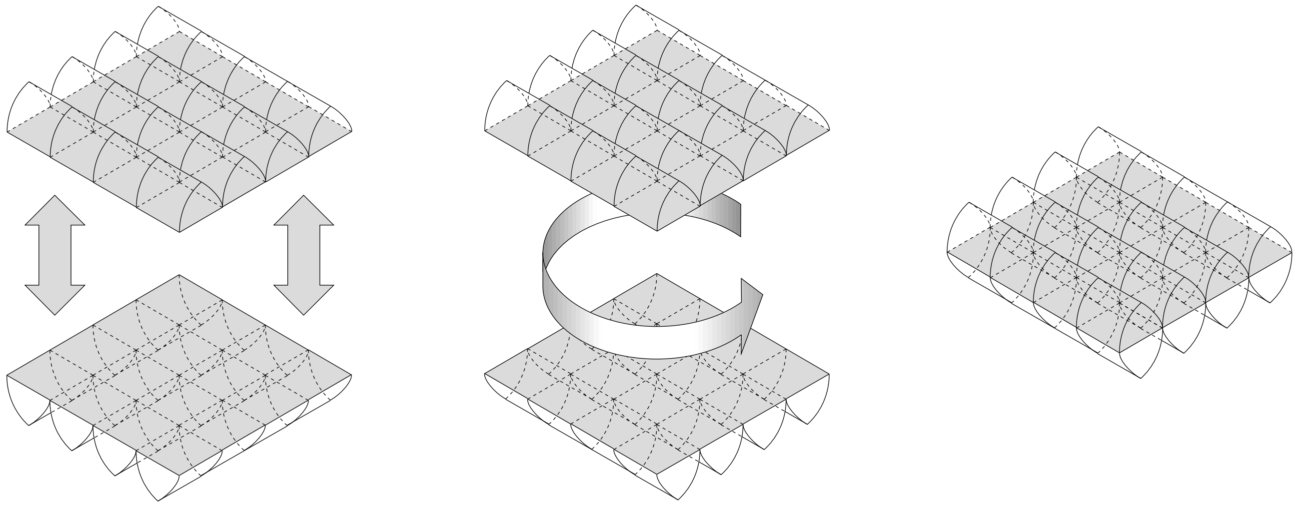


Now, arrange all sixteen tetrahedral regions of $\mathbb{S}^3$, so that they are just touching along the edges of the sections of the equatorial torus which divide them. Arranged in this way, we have another picture or "map" of $\mathbb{S}^3$. If we were to pull the four ridges along the top of this object together into a single line, and pull the four ridges along the bottom together into a single line, we would form the two centerlines of the Clifford Tori. Then if we glued the edges of the equatorial torus, we would wrap this object up into $\mathbb{S}^3$.

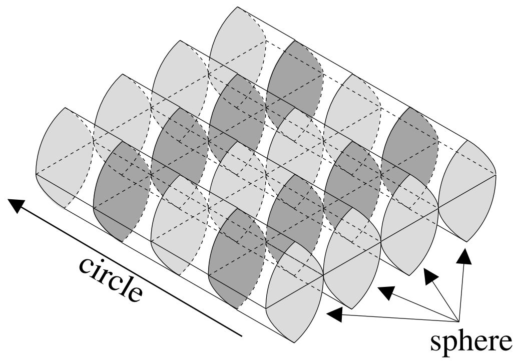


If we separate these sixteen tetrahedra into two 2×4 groups of eight tetrahedra each, we can collapse each section into one of two spheres, each with half of the equatorial torus inside, similarly to what we see in section 4.1. If we separate the sixteen tetrahedra along the surface of the equatorial torus, we can wrap each half up into a filled torus, showing that $\mathbb{S}^3$ can be thought of as two filled tori, as in section 6.1.1.

If we separate the sixteen tetrahedra along the surface of the Clifford Tori, rotate one by $\frac{\pi}{2}$, and then reglue the tetrahedra, we have actually performed the surgery on the unknot mentioned in section 6.



This new manifold appears to be made up of many spheres (here colored alternating shades of gray), arranged in a circle. In other words, it is $\mathbb{S}^2 \times \mathbb{S}^1$. Here is a picture with five different spheres colored in (Actually, the last one and the first one are the same sphere).



8 Conclusion

So we see that the sphere is a fascinating object. There several general characteristics of the infinite family of $\mathbb{S}^n$. Stereographic projection, the two balls method, and the triangulation method all generalize to $\mathbb{S}^n$.

There are also some unique structures on $\mathbb{S}^3$ which do not occur on other spheres, like the symmetrical division into two tori. And finally, there is this construction, mentioned in section 7, which takes advantage of both unique and general characteristics of $\mathbb{S}^3$ and encapsulates them in a single object.

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